Journal of Approximation Theory **113**, 165–171 (2001)
doi:10.1006/jath.2001.3585, available online at http://www.idealibrary.com on IDEAL®

Some Remarks on Equivalence of Moduli of Smoothness[1]

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Communicated by Zeev Ditzian

Received October 21, 1999; accepted in revised form April 9, 2001

The present paper investigates polynomials for which the inverse inequality for moduli of smoothness holds. The technique for approach is different from the previous works for splines and is elegantly organized.

Key Words: equivalence; modulus of smoothness; polynomial.

Let $f(x)$ be a continuous function on the interval $[a, b]$ which has m continuous derivatives, in symbol, $f \in C^m_{[a,b]}$ $(C_{[a,b]} = C^0_{[a,b]})$, and $\omega_k(f, t)_{[a,b]}$ be the modulus of smoothness of order k of $f \in C_{[a,b]}$, as usual. We will write $\omega(f, t) = \omega(f, t)_{[a,b]}$ for convenience if there is no confusion.

It is well known that $\omega_m(f, t) \leqslant t^k \omega_{m-k}(f^{(k)}, t)$ for $m \geqslant k$ if $f \in C^m_{[a,b]}$, where $\omega_0(f, t) = \|f\|_{[a,b]} := \max_{a \leqslant x \leqslant b} |f(x)|$.

The inverse result of the above inequality does not hold in general. However, for some functions $f \in C_{[a,b]}$, one has

$$t^k \omega_{m-k}(f^{(k)}, t) \leqslant C\omega_m(f, t) \tag{1}$$

[1] The Research Project of The Mathematical Institute of Ningbo University. Supported in part by Zhejiang Provincial Natural Science Foundation, Ningbo Key Doctoral Funds, and by the State Key Laboratory of Southwest Institute of Petroleum, China.

0021-9045/01 $35.00

for $m \geqslant k$, where $C > 0$ is some constant independent of t for small t. This kind of works began from a result of Yu and Zhou [5], and was investigated by Hu [2] and Hu and Yu [3]. As a whole, all these results indicate that for splines with arbitrary (fixed) knots, the inequality (1) holds in general L^p spaces for small t.

The present paper will investigate polynomials for which the inequality (1) holds.

As we know (see Stechkin [4]), for trigonometric polynomials of degree n (we denote all trigonometric polynomials of degree n by T_n), the following inequality holds:

THEOREM 1. *Let $f \in T_n$, $m \geqslant 1$, $n \geqslant 1$. Then for any $h \in [0, \pi/n]$ we have*

$$\|f^{(m)}\|_{[0, 2\pi]} \leqslant \left(\frac{n}{2 \sin nh}\right)^m \|\Delta_{2h}^m f\|_{[0, 2\pi]},$$

where $\Delta_h^m f(x)$ is the mth difference of $f(x)$ with step h.

From Theorem 1, we can easily deduce the following

THEOREM 1′. *Let $f \in T_n$, $m \geqslant 1$, $n \geqslant 1$. Then for any $0 < t \leqslant \pi/n$ and $k \leqslant m$ we have*

$$t^k \omega_{m-k}(f^{(k)}, t) \leqslant C(m)\, \omega_m(f, t),$$

where $C(m)$ is a positive constant only depending upon m.

We are going to establish an analogue for algebraic polynomials. It is clear that this as well as the following Theorem 2′ is not a direct consequence from Theorem 1 just by a simple variable change $x = \cos \theta$ since the general differences or moduli of smoothness are related to.

Let Π_n be the class of all algebraic polynomials of degree n.

THEOREM 2. *Let $f \in \Pi_{n+m}$, $m \geqslant 1$, $n \geqslant 1$. Then there is a constant $M_m > 0$ only depending upon m such that for any $h \in [0, M_m n^{-2}]$ we have*

$$h^m \|f^{(m)}\|_{[-1, 1]} \leqslant C(m)\, \|\Delta_h^m f\|_{[-1, 1-mh]}.$$

Let $T_n(x) = \cos(n \operatorname{arc} \cos x)$ be the Chebyshev polynomial of degree n, and $\xi_k = \cos(k\pi/n)$, $k = 0, 1, ..., n$, its extremum points.

LEMMA 3. *Let $f \in \Pi_n$, $f(x_0) = \|f\|_{[-1, 1]}$, $x_0 \in [\xi_{j_0+1}, \xi_{j_0}]$ for some $j_0 \in \{0, 1, ..., n-1\}$. Then*

$$f(x) \geqslant \begin{cases} \|f\|_{[-1,1]}\, \sigma_1 T_n(x), & \\ \qquad x_0 = \xi_{j_0} \quad \textit{or} \quad x_0 = \xi_{j_0+1}, \quad x \in [\xi_{j_0+1}, \xi_{j_0}], & \\ \|f\|_{[-1,1]}\, \sigma_2 \bar{T}_n(x), & \\ \qquad \textit{otherwise and} \quad x_0 \geqslant 0, \quad x \in [s_{j_0+1}, s_{j_0}], & \\ \|f\|_{[-1,1]}\, \sigma_3 \tilde{T}_n(x), & \\ \qquad \textit{otherwise and} \quad x_0 < 0, \quad x \in [s'_{j_0+1}, s'_{j_0}], & \end{cases} \tag{2}$$

where $\sigma_1 = \operatorname{sgn} T_n(\xi_{j_0})$ *for* $x_0 = \xi_{j_0}$, *or* $\sigma_1 = \operatorname{sgn} T_n(\xi_{j_0+1})$ *for* $x_0 = \xi_{j_0+1}$,

$$\bar{T}_n(x) = T_n(t), \qquad t = \frac{1+\xi_{j_0}}{1+x_0}(x-x_0)+\xi_{j_0}, \qquad \sigma_2 = \operatorname{sgn} T_n(\xi_{j_0}),$$

$$s_k = \frac{1+x_0}{1+\xi_{j_0}}(\xi_k-\xi_{j_0})+x_0, \qquad k = 0, 1, ..., n,$$

and

$$\tilde{T}_n(x) = T_n(u), \qquad u = \frac{1-\xi_{j_0+1}}{1-x_0}(x-x_0)+\xi_{j_0+1}, \qquad \sigma_3 = \operatorname{sgn} T_n(\xi_{j_0+1}),$$

$$s'_k = \frac{1-x_0}{1-\xi_{j_0+1}}(\xi_k-\xi_{j_0+1})+x_0, \qquad k = 0, 1, ..., n.$$

Proof. We only need to prove Lemma 3 for $n \geqslant 2$. When $x_0 = \xi_{j_0+1} = -1$ or $x_0 = \xi_{j_0} = 1$, the argument is similar, we only deal with the second case $x_0 = \xi_{j_0} = 1$. Set

$$\psi_n(x) = f(x) - \|f\|_{[-1,1]}\, T_n(x),$$

and assume (2) fails. Then there is an $x_1 \in (\xi_1, 1)$ such that $\psi_n(x_1) < 0$. One should note that $\psi_n(\xi_1) \geqslant 0$ and $\psi_n(1) = 0$ under this situation, hence $x_1 \neq \xi_1$ and $x_1 \neq 1$. We see that $(-1)^{k+1} \operatorname{sgn} \psi_n(\xi_k) \geqslant 0$, $k = 1, 2, ..., n$, and $\psi_n(x_1) < 0$, so that $\psi_n(x)$ has n zeros between $[-1, x_1]$, and one more zero at $\xi_0 = 1$. This contradicts the fact that any polynomial of degree n has at most n zeros.

When $x_0 = \xi_{j_0+1}$ or $x_0 = \xi_{j_0}$ but $x_0 \neq \pm 1$, the similar argument can be applied to find n zeros of

$$\psi_n(x) = f(x) - \|f\|_{[-1,1]}\, \sigma_1 T_n(x)$$

in $[-1, 1]$, and one more zero at x_0 (the multiplicity is calculated) since x_0 is a local extremum point of both $f(x)$ and $T_n(x)$ (thus $\psi_n'(x_0)=0$). This also leads to a contradiction.

Now assume $x_0 \in (\xi_{j_0+1}, \xi_{j_0})$, and without loss of generality, assume $x_0 \geqslant 0$. Set

$$t = \frac{1+\xi_{j_0}}{1+x_0}(x-x_0)+\xi_{j_0}$$

for $x \in [-1, 1]$, and

$$s_k = \frac{1+x_0}{1+\xi_{j_0}}(\xi_k - \xi_{j_0})+x_0, \qquad k=0, 1, ..., n.$$

By noting $x_0 < \xi_{j_0}$ we have for $k=0, 1, ..., n$,

$$-1 \leqslant s_k < \frac{1+x_0}{1+\xi_{j_0}}(1-\xi_{j_0})+x_0 \leqslant 1.$$

Let

$$\bar{T}_n(x) = T_n(t).$$

Then

$$\bar{T}_n(x_0) = T_n(\xi_{j_0}) = \sigma_2 \, \|T_n\|_{[-1,1]}$$

for $\sigma_2 = \operatorname{sgn} T_n(\xi_{j_0})$. Suppose the inequality (2) fails. One has a point $x_1 \in [s_{j_0+1}, s_{j_0}]$ such that

$$f(x_1) < \|f\|_{[-1,1]} \, \sigma_2 \bar{T}_n(x_1), \tag{3}$$

where, in particular, $s_{j_0} = x_0$. One must note here that when $x = s_k$, $t = \xi_k$. So

$$\operatorname{sgn} \bar{T}_n(s_k) = (-1)^k. \tag{4}$$

Write

$$\phi_n(x) = f(x) - \sigma_2 \bar{T}_n(x) \, \|f\|_{[-1,1]}. \tag{5}$$

One also must note that $x_1 \neq s_{j_0}$ and $x_1 \neq s_{j_0+1}$ since $\phi_n(s_{j_0+1}) \geqslant 0$ and $\phi_n(s_{j_0}) = \phi_n(x_0) = 0$ hold. We check that, due to (4) and (5),

$$(-1)^{k+1} \, \sigma_2 \operatorname{sgn} \phi_n(s_k) \geqslant 0, \qquad k=0, 1, ..., n,$$

and in particular,

$$\phi_n(x_0) = \phi_n(s_{j_0}) = 0, \qquad \operatorname{sgn} \phi_n(s_{j_0+1}) \geqslant 0.$$

In case $\xi_{j_0} = 1$, with the same argument as the proof of the case $x_0 = \xi_{j_0} = 1$ (by using $\phi_n(x)$ instead of $\psi_n(x)$) we can achieve the required result. Now assume $\xi_{j_0} < 1$, we see $\phi_n'(x_0) = 0$ since x_0 is a local extremum point of both $f(x)$ and $\bar{T}_n(x)$ (this happens because ξ_{j_0} cannot be 1, and cannot be -1 due to $x \geqslant 0$ and $n \geqslant 2$). Furthermore $\phi_n(x_1) < 0$ by (3). Therefore $\phi_n(x)$ has $n - j_0 - 1$ zeros in $[s_n, s_{j_0+1}]$, has $j_0 - 1$ zeros in $[s_{j_0-1}, s_0]$, and has one zero in $[s_{j_0+1}, x_1]$. Furthermore, we see that $\phi_n(x)$ has two zeros at x_0 (the multiplicity is calculated). All together, $\phi_n(x)$ has $n+1$ zeros in $[s_n, s_0] \subset [-1, 1]$, that is impossible since $\phi_n(x)$ is a polynomial of degree n. This contradiction proves the conclusion we require. ∎

Proof of Theorem 2. We first prove the case $m = 1$. Assume $f'(x_0) = \|f'\|_{[-1,1]}$, the other case $f'(x_0) = -\|f'\|_{[-1,1]}$ can be treated similarly. Without loss of generality, with all the notations of Lemma 3, we also assume $x_0 \in [\xi_{j_0+1}, \xi_{j_0}]$, $x_0 \neq \xi_{j_0}$, $x_0 \neq \xi_{j_0+1}$, and $x_0 \geqslant 0$. For other cases mentioned in Lemma 3, we have similar arguments. By Lemma 3, for all $x \in [s_{j_0+1}, s_{j_0}]$,

$$f'(x) \geqslant \|f'\|_{[-1,1]} \, \sigma \bar{T}_n(x)$$

for $\sigma = \operatorname{sgn} T_n(\xi_{j_0})$. Note that $t = ((1+\xi_{j_0})/(1+x_0))(x-x_0) + \xi_{j_0}$, let $y_0 = x_0 = s_{j_0}$,

$$y_1 = \frac{1+x_0}{1+\xi_{j_0}} \left(\cos \frac{(j_0 + 2/3)\,\pi}{n} - \xi_{j_0} \right) + x_0,$$

we see $0 < y_1 < y_0$, and

$$\bar{T}_n(y_0) = T_n(\xi_{j_0}), \qquad \bar{T}_n(y_1) = T_n(\cos((j_0 + 2/3)\,\pi/n)).$$

For any $0 < h \leqslant y_0 - y_1$,

$$\begin{aligned} |f(y_0 - h) - f(y_0)| &= \left| \int_0^h f'(y_0 - u)\, du \right| \\ &\geqslant \|f'\|_{[-1,1]} \int_0^h |\bar{T}_n(y_0 - u)|\, du \geqslant \|f'\|_{[-1,1]} \frac{h}{2}, \end{aligned}$$

or for any $0 < h \leqslant y_0 - y_1$,

$$\|f'\|_{[-1,1]} \leqslant \frac{2}{h} \|f(x+t) - f(x)\|_{[-1,1-h]}. \tag{6}$$

It is not difficult to calculate that

$$y_0 - y_1 \geqslant \frac{1}{2}\left(\cos\frac{j_0\pi}{n} - \cos\frac{(j_0 + 2/3)\,\pi}{n}\right) \geqslant Mn^{-2},$$

where $M > 0$ is an absolute constant. Thus for any $0 < h \leqslant Mn^{-2}$, (6) holds.

When $m \geqslant 1$ and $0 < h \leqslant M_m n^{-2}$, we can reach that

$$\begin{aligned}\|\Delta_h^{m+1} f(x)\|_{[-1,1-(m+1)h]} &= \|\Delta_h^m(f(x+h) - f(x))\|_{[-1,1-(m+1)h]} \\ &\geqslant C(m)\, h^m \,\|f^{(m)}(x+h) - f^{(m)}(x)\|_{[-1,1-h]} \\ &\geqslant C(m)\, h^{m+1}\, \|f^{(m+1)}\|_{[-1,1]}\end{aligned}$$

by induction, where $M_m > 0$ is a constant only depending upon m. Up to this stage, we have finished the proof. ▮

Remark. Ditzian *et al.* [1] give a similar inequality on algebraic polynomials in terms of $\varphi(x) = \sqrt{1 - x^2}$:

$$h^m \,\|\varphi^m P_n^{(m)}\|_{[-1,1]} \leqslant C(m)\, \|\Delta_{h\varphi}^m P_n\|_{[-1,1]}$$

holds for $0 \leqslant h \leqslant Cn^{-1}$. One can deduce Bernstein type inequality

$$|P_n^{(m)}(x)| \leqslant C(m)\, n^m \varphi^{-m}(x)\, \|P_n\|_{[-1,1]}$$

directly from their result. We note that a direct corollary from our present result is Markov inequality (except for a constant). Those two inequalities form complete inverse inequalities for the mth difference of an algebraic polynomial and its mth derivative in uniform norm.

From Theorem 2, we can immediately deduce that

THEOREM 2′. *Let $f \in \Pi_{n+m}$, $m \geqslant 1$, $n \geqslant 1$. Then for any $t \in [0, n^{-2}]$ we have*

$$t^k \omega_{m-k}(f^{(k)}, t) \leqslant C(m)\, \omega_m(f, t).$$

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